

# Introduction to Mathematical Statistics

## Introduction to Mathematical Statistics

Key concepts, definitions, and tools that form the foundation of statistical inference.

## 1. What Is Mathematical Statistics?

| Aspect                               | Description                                                                                                                                                                                                                                                                      |
|--------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Goal                                 | Use probability theory to model data-generating processes and to draw conclusions (estimation, testing, prediction). <ul style="list-style-type: none"><li>• Probability models (distributions, random variables)</li><li>• Sampling theory (iid samples, convergence)</li></ul> |
| Scope                                | <ul style="list-style-type: none"><li>• Estimation theory (point, interval)</li><li>• Hypothesis testing</li><li>• Asymptotic methods (large-sample approximations)</li></ul>                                                                                                    |
| Contrast with Descriptive Statistics | Descriptive statistics summarises observed data; mathematical statistics provides the <i>theoretical</i> justification for those summaries and for decisions based on data.                                                                                                      |

## 2. Probability Foundations

### 2.1 Random Variables & Distributions

- **Random variable (RV):** measurable function  $(X:\Omega\rightarrow\mathbb{R})$ .
- **Distribution function**  $(F_X(x)=P(X\leq x))$ .
- **Probability density/mass function** (pdf/pmf)  $(f_X(x)=F'_X(x))$  (continuous) or  $(p_X(x)=P(X=x))$  (discrete).

### 2.2 Expectation & Moments

- **Expectation:**  $(\mathbb{E}[X]=\int x f_X(x) dx)$  (continuous) or  $(\sum x p_X(x))$  (discrete).
- **Variance:**  $(\text{Var}(X)=\mathbb{E}[(X-\mu)^2])$ .
- **Higher moments:**  $(\mu'_k=\mathbb{E}[X^k])$ , **central moments**  $(\mu_k=\mathbb{E}[(X-\mu)^k])$ .

# Introduction to Mathematical Statistics

## 2.3 Common Families

| Family                                            | Typical Parameters                   | Key Properties                      |
|---------------------------------------------------|--------------------------------------|-------------------------------------|
| <b>Bernoulli</b> ( $B(p)$ )                       | $(p \in (0,1))$                      | $(\mu=p, \sigma^2=p(1-p))$          |
| <b>Binomial</b> ( $\text{Bin}(n,p)$ )             | $(n \in \mathbb{N}, p)$              | Sum of $(n)$ iid Bernoulli          |
| <b>Poisson</b> ( $\text{Pois}(\lambda)$ )         | $(\lambda > 0)$                      | Mean = variance = $(\lambda)$       |
| <b>Normal</b> ( $N(\mu, \sigma^2)$ )              | $(\mu \in \mathbb{R}, \sigma^2 > 0)$ | Closed under linear transformations |
| <b>Exponential</b> ( $\text{Exp}(\lambda)$ )      | $(\lambda > 0)$                      | Memoryless property                 |
| <b>Gamma</b><br>( $\text{Gamma}(\alpha, \beta)$ ) | shape $(\alpha)$ , rate $(\beta)$    | Generalises exponential, chi-square |

---

## 3. Sampling Distributions

### 3.1 Random Samples

- **Sample:**  $(X_1, \dots, X_n)$  i.i.d. from distribution  $(F)$ .
- **Sample statistics** (functions of the data):  $(\bar{X}, S^2, \max X_i, \dots)$  etc.

### 3.2 Important Sampling Distributions

| Statistic               | Distribution (under normality) | Use                  |
|-------------------------|--------------------------------|----------------------|
| Sample mean $(\bar{X})$ | $(N(\mu, \sigma^2/n))$         | Point estimation, CI |
| Sample variance $(S^2)$ | $(\sigma^2)$                   |                      |