

Introduction to Topology

Introduction to Topology – Study Notes

1. What Is Topology?

- **Definition:** Topology is the branch of mathematics that studies properties of spaces that are preserved under continuous deformations (stretching, bending, but not tearing or gluing).
 - **Intuition:** Think of a coffee cup and a doughnut; topologically they are the same because each has one “hole”.
 - **Goal:** Classify spaces by their *topological invariants* (properties that do not change under homeomorphisms).
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2. Basic Concepts

Concept	Description	Example
Set	A collection of points (elements).	$(X = \{1, 2, 3\})$
Topology on a set	A collection $(\mathcal{T})$ of subsets of (X) satisfying: 1. $(\emptyset, X \in \mathcal{T})$ 2. Arbitrary unions of members of $(\mathcal{T})$ are in $(\mathcal{T})$ 3. Finite intersections of members of $(\mathcal{T})$ are in $(\mathcal{T})$.	Standard topology on $(\mathbb{R})$: all open intervals $((a, b))$.
Open set	Any member of the topology $(\mathcal{T})$.	$((0, 1))$ in $(\mathbb{R})$.
Closed set	Complement of an open set.	$[0, 1]$ is closed in $(\mathbb{R})$.
Neighbourhood	Any set containing an open set around a point.	For $(x=2)$ in $(\mathbb{R})$, the interval $((1.5, 2.5))$ is a neighbourhood.
Basis	A subcollection $(\mathcal{B})$ of $(\mathcal{T})$ such that every open set is a union of basis elements.	In $(\mathbb{R})$, the set of all open intervals with rational endpoints forms a countable basis.

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Subspace topology	If $(Y \subseteq X)$, the topology on (Y) is $(\{U \cap Y \mid U \in \mathcal{T}\})$.	The interval $([0,1])$ inherits the usual topology from $(\mathbb{R})$.
Product topology	For spaces (X) and (Y) , the topology on $(X \times Y)$ generated by products of open sets $(U \times V)$.	$(\mathbb{R}^2)$ with the Euclidean topology is a product of two copies of $(\mathbb{R})$.
Quotient topology	Given a surjection $(q: X \rightarrow Y)$, a set $(U \subseteq Y)$ is open iff $(q^{-1}(U))$ is open in (X) .	Collapsing the boundary of a disk to a point yields a sphere.

3. Continuous Maps

- **Definition:** $(f: X \rightarrow Y)$ is **continuous** if for every open set $(V \subseteq Y)$, the pre-image $(f^{-1}(V))$ is open in (X) .
- **Equivalently:** Pre-images of closed sets are closed.
- **Homeomorphism:** A bijective continuous map whose inverse is also continuous. Two spaces related by a home